The Differential Equations Satisfied by Abelian Theta Functions of Genus Three.

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In several papers* and in his book "Multiply Periodic Functions,"† Baker has given the differential equations satisfied by hyperelliptic theta functions. His method is most satisfactory in its final outcome, because the constants that occur in the equations are expressed in terms of the associated Riemann Surface, with definitely known cross-cuts, but owing to this complete determination of the equations the process involves long and complicated algebraic manipulation. Its application to any but the hyperelliptic functions would seem almost impossible. Now if we start from the general definition of a theta function as a uniform integral function of several variables, that possesses certain period properties, we can discover enough about its nature to enable us to give the general forms of the differential equations it satisfies, and then it appears that conditions of coexistence of these equations are sufficient to make them precise.

For example, the general nature of the theta functions of genus 2 leads us to the conclusion that it must satisfy five differential equations of the fourth order, of a certain particular form. These equations involve twenty constants, but conditions to be satisfied in order that they may coexist reduces this number, so that finally the equations depend on only three essential constants, and therefore we conclude that the general solution of the final differential equations must be such a theta function. It is possible that, the equations once obtained, they may be integrated directly, and thus the theory may be made complete from this point of view.‡

The purpose of this paper is to determine the differential equations whose general solution is a general theta function of genus 3. As a first illustration,

* *Proc. Camb. Phil. Soc.*, Vol. IX (1898), p. 517; *ibid.*, Vol. XII (1903), p. 219; *Acta Math.*, t. XXVII (1903), p. 135.

† Cambridge University Press (1907).

‡ Cf. "*Multiply Periodic Functions*," p. 44 sqq.

we propose to apply the method outlined to the case of $p=2$ to obtain Baker's results.

The case of $p=3$ leads to a division of the types of equation into two classes. The first of these turns out to be the hyperelliptic case. By adding a suitable exponential factor to the theta function the equations are given by means of covariants of certain ternary forms; these forms are: 1) a quadratic whose coefficients are the second derivatives of the logarithm of the theta function; 2) a cubic whose coefficients are the third derivatives; 3) a quartic whose coefficients are the fourth derivatives, and similarly for higher derivatives, and 4) certain fixed forms. For the hyperelliptic case the fixed forms are a conic and a quartic. These two curves determine a binary octavic, namely that cut out on the conic by the quartic, and this case is thus associated with the invariant and covariant properties of a binary octavic. In the non-hyperelliptic case there is only one fixed form, a general quartic. It thus appears that this case is closely connected with the geometrical properties of a general quartic. This is interesting in view of the fact that a non-hyperelliptic curve of genus three can always be birationally transformed into a non-singular quartic, whereas this is not true of a hyperelliptic curve of genus three, for which the reduced curve of lowest order is a quintic with a triple point.

§1.

In the subsequent work we need some general definitions and theorems, which we quote from Baker's "Abelian Functions." *

Suppose that we have four matrices $\omega, \omega', \eta, \eta'$, each of p rows and columns, which satisfy the conditions: 1) that the determinant of ω is not zero; 2) that the matrix $\omega^{-1}\omega'(\equiv\tau)$ is symmetrical; 3) that for real values of $n_1, n_2, \ldots, n_p$ the quadratic form $\omega^{-1}\omega' n^2$ has its imaginary part positive; 4) that the matrix $\eta\omega^{-1}$ is symmetrical; 5) that $\eta' = \eta\omega^{-1}\omega' - \frac{1}{2}\pi i\bar{\omega}^{-1}$. We put

$$a = \tfrac{1}{2}\eta\omega^{-1}, \quad h = \tfrac{1}{2}\pi i\omega^{-1}, \quad b = \pi i\omega^{-1}\omega',$$

so that

$$\eta = 2a\omega, \quad \eta' = 2a\omega' - \bar{h}, \quad h\omega = \tfrac{1}{2}\pi i, \quad h\omega' = \tfrac{1}{2}b;$$

and we write

$$\lambda_m(u) = H_m(u + \tfrac{1}{2}\Omega_m) - \pi i m m',$$

where

$$H_m = 2\eta m + 2\eta' m', \quad \Omega_m = 2\omega m + 2\omega' m'.$$

* Hereafter quoted as A. F.

Also let Q, Q' denote two assigned rows of p rational quantities, and suppose $\Pi(u)$ to be an integral function of the p arguments $u_1, u_2, \ldots, u_p$ that satisfies the equation

$$\Pi(u+\Omega_m)=e^{r\lambda_m(u)+2\pi i(mQ'-m'Q)}\Pi(u)$$

for all integral values of m, m'. Then the function $\Pi(u)$ is called a theta function of order r, with the associated constants 2ω, $2\omega'$, 2η, $2\eta'$, and the characteristic (Q, Q'). [A. F. 447, 448.]

It may be proved that the function $\Pi(u)$ exists, and further that if the associated constants and the characteristic are given, there are not more than r^p such functions linearly independent of one another. [A. F. 448–452.]

We notice that the essential character of $\Pi(u)$ is unchanged if a linear transformation be made on the variables u, or if it is multiplied by an exponential factor of the type e^{au^2}, where a is a symmetrical matrix.

The limit to the number of linearly independent functions of order r may be reduced if $\Pi(u)$ is an even function or an odd function of its arguments taken together. In this case it is not difficult to prove that the constants Q, Q' must be half integers [A. F. 462], and the results are:

If $\Pi(-u)=\varepsilon\Pi(u)$, where $\varepsilon=\pm 1$, and r is even, whilst (Q, Q') consists of integers, the number of linearly independent functions $\Pi(u)$ is

$$\leq \tfrac{1}{2}r^p+2^{p-1}\varepsilon.$$

When r is odd, or when r is even and the characteristic (Q, Q') does not consist wholly of integers, then the number of linearly independent functions is

$$\leq \tfrac{1}{2}r^p+\tfrac{1}{4}[1-(-1)^r]\varepsilon e^{4\pi i QQ'}. \quad \text{[A. F. 463.]}$$

Now suppose θ to be a function of the first order, with half-integer characteristic (q, q'). Then from the properties of such functions we have the result that $e^{4\pi i qq'}=\varepsilon$. [A. F. 251.]

If $\Pi(u)$ is θ^r, it is clear that Π is of the r-th order, with characteristic (rq, rq'). Hence, for functions with the same defining properties as θ^r the above numbers become $\frac{1}{2}r^p+2^{p-1}$ if r is even, and $\frac{1}{2}(r^p+1)$ if r is odd.

In particular we note that if $r=2$, $\theta(u+v)\theta(u-v)$ has the same defining properties as θ^2, and hence there must be a linear relation, with coefficients independent of u, connecting 2^p+1 of these functions for 2^p+1 different values of the arguments v. There is a similar result for functions of the type $\theta(u-v)\,\theta(u-w)\,\theta(u+v+w)$ when $r=3$, and so on for higher values of r.

Now if $F(u)$ is a function multiply periodic in ω, ω', *i. e.*, one such that

$$F(u+\Omega_m)=F(u),$$

for all values of the integers m, m', and if it is made integral by being multiplied by some power, s, of θ, it is clear that if $r>s$, $\theta^r F(u)$ has the same defining properties as θ^r, and hence there is a linear relation among either $\frac{1}{2}r^p+2^{p-1}$ or $\frac{1}{2}(r^p+1)$ such functions, according as r is even or odd, provided $F(u)$ is even.

The second derivatives of $\log\theta$ are such multiply periodic functions. BAKER, BOLZA and others use the notation $\wp_{ij}(u)=-\frac{\partial^2}{\partial u_i\,\partial u_j}(\log\theta)$. [See, *e. g.*, A. F. 292, etc.] We shall find it convenient to use (ij) for this function, and similarly, in general,

$$-\frac{\partial^k}{\partial u_r\,\partial u_s\,\partial u_t\ldots.}(\log\theta)\text{ is written }(rst\ldots.)\,;$$

and $(rst\ldots.)$ is a multiply periodic function, which is made integral on multiplication by θ^k.

We shall assume in the remainder of this paper that there is no polynomial relation of either the first or second order connecting the derivatives (ij). Such a relation would in fact be a limitation on the generality of the constants in ω, ω'. For example, if $p=2$, we could reduce a linear relation among (11), (12), (22) to the form $(12)=0$, and θ would reduce to the product of two elliptic theta functions. The particular cases for which such a relation exists are of some interest when $p=3$. We propose to consider them in a later paper.

§ 2.

We consider first the case $p=2$. In this case there are four linearly independent functions of the second order with the same period properties as θ^2. They are θ^2, $\theta^2(11)$, $\theta^2(12)$, $\theta^2(22)$. It is easy to verify that $[(11)(22)-(12)^2]\,\theta^3$ is integral, and hence the five functions with the same period properties as θ^3 are

$$\theta^3\{[(11)(22)-(12)^2],\ (11),\ (12),\ (22),\ 1\}.$$

To save repetition we shall say that a function $f(u)$ is of the r-th order when $f(u)$ is even, multiply periodic, and $\theta^r f(u)$ is integral. It is clear that a function of the r-th order is also a function of the s-th order if r is less than s.

The functions of the fourth order, ten in number, are

$$1,\ (11),\ (12),\ (22), \text{ and all products of the type } (pq)(rs).$$

Of the sixth-order functions twenty are linearly independent. Now they include I) $[(11)(22)-(12)]^2$, II) all products $(pq)(rs)(tu)$, III) the functions of the fourth order.

These are in number 21, and therefore they must be connected by a linear relation. We thus see *a priori* that there is a quartic relation among the three second derivatives (11), (12), (22). This turns out to be Kummer's Quartic Surface.

Again (pqr) is of the third order, except that it is an odd function, and hence any product $(pqr)(stu)$, being of the sixth order, must be expressible as a cubic polynomial in (11), (12), (22). These considerations are useful as showing the kind of relations we are to expect. To obtain them we make use of the fact that

$$\frac{\theta(u-v)\,\theta(u+v)}{\theta^2(u)}$$

is a function of the second order for all values of the variables v, and hence if it is expanded in powers of the v's, all its coefficients are such functions. We write $\theta = e^{-f}$; then

$$\frac{\theta(u-v)\,\theta(u+v)}{\theta^2(u)} = exp\left\{-2\left[\frac{1}{\underline{|2}}\left(v\frac{\partial}{\partial u}\right)^2 f + \frac{1}{\underline{|4}}\left(v\frac{\partial}{\partial u}\right)^4 f + \dots\right]\right\}.$$

If the right-hand side be expanded, the coefficients of products of the fourth and sixth orders of v are readily obtained, and we have

$$(pqrs) - 2[(pq)(rs) + (pr)(qs) + (ps)(qr)],$$
$$(pqrstu) - 2\Sigma(pq)(rstu) + 4\Sigma(pq)(rs)(tu),$$

for these coefficients, where the summations extend to all possible combinations of the six letters p, q, r, s, t, u. These expressions are therefore both of them functions of the second order.

Hence we must have $(pqrs) - 2[(pq)(rs) + (pr)(qs) + (ps)(qr)] =$ a linear function of 1, (11), (12), (22), $= \sum_{h,k} b_{pqrs}^{(hk)}(hk) + b_{pqrs}$, say, where the b's are constants, and the summation extends once to each pair of values of h, k. Now, by giving p, q, r, s the values 1, 2 we obtain five such equations. These are differential equations of the fourth order for a single function f, and their coexistence by no means follows for general values of the constants b.

In fact, if we differentiate the five equations once, and eliminate fifth derivatives, we obtain four equations among third and second derivatives; these are homogeneous and linear in the third derivatives, which may therefore be eliminated. We thus have an equation among second derivatives only. We might use this equation to obtain by differentiation other homogeneous linear equations in third derivatives, and then by elimination other equations connecting the second derivatives. It is simpler, however, to differentiate the five fundamental differential equations twice and then to eliminate from them the sixth derivatives that occur. We thus get eight equations involving second derivatives and certain functions of third derivatives of the type $(pqr)(hks) - (pqs)(hkr)$. There are only three of these latter functions, and thus by elimination we get five equations which turn out to be of the form

$$A_i[(11)(22)-(12)^2] + B_i(11) + C_i(12) + D_i(22) + E_i = 0 \quad (i = 1, 2, \ldots, 5),$$

where A, B, C, D, E are constants. As we have assumed that no such relation exists, A, B, C, D, E must all be zero.

If we denote the right-hand side of the typical fundamental equation by B_{pqrs}, and additional suffixes denote differentiations, and if $B_{pqr\alpha,\, s\beta} - 6(s\beta)B_{pqr\alpha}$, after fourth derivatives have been replaced by their values in terms of second derivatives, is written $[pqr\alpha,\, s\beta]$, it is not difficult to see that a typical one of the equations just mentioned is

$$\begin{aligned}[pqr\alpha,\, s\beta] + [pq\alpha s,\, r\beta] + [p\alpha rs,\, q\beta] + [\alpha qrs,\, p\beta] \\ = [pqr\beta,\, s\alpha] + [pq\beta s,\, r\alpha] + [p\beta rs,\, q\alpha] + [\beta qrs,\, p\alpha].\end{aligned}$$

The five equations are therefore

$$[1111,\, 12] = [1112,\, 11], \quad [1112,\, 22] = [1222,\, 11],$$
$$[1111,\, 22] + 2[1112,\, 12] = 3[1122,\, 11],$$

and two similar equations obtained by interchanging 1 and 2. We take the second-degree terms in these equations first. $[1111,\, 12] = [1112,\, 11]$ becomes, on expansion,

$$(2b_{1111}^{(12)} + 4b_{1112}^{(22)})[(11)(22)-(12)^2] + \text{linear terms} = 0.$$

Thus we have $b_{1111}^{(12)} + 2b_{1112}^{(22)} = 0$.

Similarly, from the second equation we have $b_{1112}^{(11)} = b_{1222}^{(22)}$, and from the third $b_{1111}^{(11)} - b_{1112}^{(12)} = 3b_{1122}^{(22)}$.

We thus have five relations among the constants b. If these are satisfied, it appears that the remaining relations obtained from the above five equations determine the values of the constants b_{pqrs} uniquely, and otherwise lead to no new relations.

Now, the multiplication of θ by an exponential factor e^{au^2} does not alter its essential nature, and is equivalent to giving arbitrary additive constants to the derivatives (11), (12), (22). If we thus modify our equations, we may make

$$2b_{1111}^{(11)}+b_{1112}^{(12)}=0,\quad 2b_{2222}^{(22)}+b_{1222}^{(12)}=0,\quad 2b_{1112}^{(11)}+b_{1122}^{(22)}=0,$$

and then the fundamental equations take precisely the form given by Baker, "Multiply Periodic Functions," p. 49.

We can now verify without trouble that the equations are compatible, obtain the Kummer and Weddle Surfaces, and obtain the relations for products of third derivatives, exactly as in Baker. We are, however, using this case as an illustration of the method for $p=3$, and hence shall indicate in outline another method for getting $(pqr)(stu)$ and the Kummer Surface.

We use Baker's notation for the coefficients of the differential equations for fourth derivatives, so that

$$\begin{aligned}
-\tfrac{1}{3}B_{1111}&=a_0a_4-4a_1a_3+3a_2^2+a_2(11)-2a_1(12)+a_0(22),\\
-\tfrac{1}{3}B_{1112}&=\tfrac{1}{2}(a_0a_5-3a_1a_4+2a_2a_3)+a_3(11)-2a_2(12)+a_1(22),\\
-\tfrac{1}{3}B_{1122}&=\tfrac{1}{6}(a_0a_6-9a_2a_4+8a_3^2)+a_4(11)-2a_3(12)+a_2(22),\\
-\tfrac{1}{3}B_{1222}&=\tfrac{1}{2}(a_1a_6-3a_2a_5+2a_3a_4)+a_5(11)-2a_4(12)+a_3(22),\\
-\tfrac{1}{3}B_{2222}&=a_2a_6-4a_3a_5+3a_4^2+a_6(11)-2a_5(12)+a_4(22).
\end{aligned}$$

It is interesting to notice the covariantive form of the equations. If we take du_1 and du_2 as variables x_1 and x_2, then any linear transformation on the u's is equivalent to the same linear transformation on the x's; it is clear that $d^4\wp$ and $d^2\wp$,

$$\equiv(11)x_1^2+2(12)x_1x_2+(22)x_2^2,\quad \equiv a_x^2\equiv b_x^2\equiv c_x^2\equiv\ldots,$$

are invariant under such a transformation. Also

$$(a_0,\,a_1,\,a_2,\,\ldots,\,a_6)(x_1x_2)^6\equiv\alpha_x^6\equiv\beta_x^6\equiv\ldots$$

is an associated sextic, and our fundamental equations are given by

$$d^4\wp-6(d^2\wp)^2=-3(\alpha a)^2\alpha_x^4-\tfrac{3}{2}(\alpha\beta)^4\alpha_x^2\beta_x^2,\ \equiv B,\ \text{say}.$$

From the coefficient, already given, of v^6 in the original expansion we have $d^6\wp-30d^2\wp\,d^4\wp+60(d^2\wp)^3=$ a homogeneous sextic in du_1, du_2, of which the

coefficients are functions of the second order. If we put this equal to X, X must have its coefficients linear functions of the second derivatives. Now $d^4\wp - 6(d^2\wp)^2 = B$, and hence on differentiation $d^6\wp - 12 d^2\wp d^4\wp - 12(d^2\wp)^2 = d^2 B$.

If we eliminate $d^6\wp$ and $d^4\wp$ from these equations, we have

$$12\{(d^3\wp)^2 - 4(d^2\wp)^3\} + d^2B - 18Bd^2\wp = X.$$

Now we have already indicated the method of obtaining expressions $(pqr)(hks) - (pqs)(hkr)$, and therefore we can obtain the various products $(pqr)(hks)$ by equating cofficients of powers and products of du_1, du_2, in the above equation, as soon as we know the coefficients of X. For example,

$$\begin{aligned}(111)^2 - 4(11)^3 + 3[a_2(11) - 2a_1(12) + a_0(22)](11)& \\ + a_0[(11)(22) - (12)^2] &= \lambda(11) + \mu(12) + \nu(22) + \rho,\end{aligned}$$

where λ, μ, ν, ρ are constants to be determined.

There are two methods for determining the unknown constants. In the first method we differentiate twice and eliminate fourth and fifth derivatives by means of the fundamental fourth-order equations. We also substitute for the various squares and products of third derivatives that occur. We then remain with an equation which turns out to be in some cases of the second degree, in others of the fourth degree in second derivatives. Now those of the second degree must vanish identically, and by equating their coefficients to zero we have enough equations to determine the unknown constants. If these are substituted in the fourth-degree equations obtained, they all reduce to one and the same equation, which is that of Kummer's Quartic Surface.

In the second method we differentiate the fundamental equations, and by subtraction eliminate fifth derivatives. We thus remain with an equation linear in third derivatives. Such an equation, for example, is

$$6(12)(111) - 6(11)(112) = 3a_3(111) - 9a_2(112) + 9a_1(122) - 3a_0(222).$$

We multiply these equations by the various third derivatives, and substitute for the squares and products of third derivatives their expressions in terms of second derivatives. As before, we get equations either of the second or of the fourth degree in second derivatives, and again we can determine the unknown constants and obtain Kummer's Quartic. In either case the work may be somewhat simplified, if we notice that $d^2B - 6Bd^2\wp = 4\alpha_x^6[(11)(22) - (12)^2] +$ a quantity

linear in second derivatives, by incorporating this linear quantity into X. The final result in symbolic form is

$$(d^3\wp)^2 - 4(d^2\wp)^3 = -\tfrac{1}{2}(ab)^2 a_x^6 - 3(\alpha a)^2 \alpha_x^4 b_x^2 - 9(\alpha a)^2(\alpha\beta)^2 \alpha_x^2 \beta_x^4 + \tfrac{9}{2}(\alpha a)(\beta a)\alpha_x^3 \beta_x^3 + \tfrac{27}{8}(\alpha\beta)^2(\beta\gamma)^2(\gamma\alpha)^2 \alpha_x^2 \beta_x^2 \gamma_x^2.$$

The Kummer Quartic is an invariant of a_x^6 and α_x^2. Its symbolic expression is

$$4(ab)^2(cd)^2 - 16(\alpha a)^2(ab)^2(ac)^2 + 6(\alpha a)^2(\beta b)^2(\alpha\beta)^4 + (ab)^2(\alpha\beta)^6 + 9(\alpha a)^2(\alpha\beta)^2(\alpha\gamma)^2(\beta\gamma)^4 - \tfrac{9}{8}(\beta\gamma)^2(\gamma\alpha)^2(\alpha\beta)^2(\alpha\delta)^2(\beta\delta)^2(\gamma\delta)^2 = 0.$$

If we write $d^3\wp, \equiv (111)x_1^2 + \ldots, \equiv p_x^3 \equiv q_x^3 \equiv r_x^3 \equiv \ldots,$ we have for the symbolic equation of the Weddle Surface

$$(hp)^2(hq)(pq)(qr)^2 = 0 \ \ [\text{where } h_x^3 \equiv (\alpha p)^3 \alpha_x^3].$$

In the above work we have assumed that there exists no quadratic relation among (11), (12), (22); it is interesting to note that the assumption of the existence of such a relation leads to a linear relation among these quantities. By a proper choice of variables such a linear relation could be reduced either to $(11) = 0$, or to $(12) = 0$.

The latter shows that θ must be the product of two elliptic θ functions, whilst the former implies that θ is the product of an elliptic θ and an exponential, e^{Au_1+B}, where A and B are constants.

§ 3.

We now consider the case of $p = 3$. We shall find it convenient to use Δ to denote the determinant

$$\begin{vmatrix} (11), & (12), & (13) \\ (21), & (22), & (23) \\ (31), & (32), & (33) \end{vmatrix}$$

and Δ_{rs} to denote the cofactor of (rs) in Δ.

It may be verified without difficulty that Δ, though apparently of the sixth, is really of the fourth order, whilst Δ_{rs} is of the third order.

In this case there are eight functions of the second order. Of these we have seven, the quantities 1, (11), (12), (13), (22), (23), (33). There must be one linearly independent of these, and this one we call Y. It is to be noticed that for simplifying our equations we may modify Y by adding to it any linear function of the known second order functions (pq) and 1.

The number of linearly independent third-order functions is $\frac{1}{2}(3^3+1), =14$. We have the eight already given, and the six functions Δ_{rs}. We see at once that there are two cases according as these are or are not linearly independent. As we are assuming that no quadratic relation exists among the quantities (pq), we see that if these 14 third-order functions are not independent, there is a relation

$$Y = \Sigma a_{pq}\Delta_{pq} + \Sigma b_{pq}(pq) + c,$$

where the a's, b's and c are constants. This turns out to be the hyperelliptic case.

There are $\frac{1}{2}4^3 + 4 = 36$ functions of the fourth order. Now we have already eight of these, namely the second-order functions. In addition we must have all products $(pq)(rs)$, $(pq)Y$, Y^2, and Δ. These are in all 37, and hence they must be connected by at least one linear relation. In the hyperelliptic case this relation is the one given above. In the other we shall show later that it may be reduced to the form

$$Y^2 + 2\Delta = \text{a quadratic in the second derivatives } (pq).$$

The number of sixth-order functions is $\frac{1}{2}6^3 + 4 = 112$. We have

I)	21 products	$\Delta_{pq}\Delta_{rs}$,
II)	56 "	$(pq)(rs)(tu)$,
III)	21 "	$Y(pq)(rs)$,
IV)	6 "	$Y^2(pq)$,
V)	21 "	$(pq)(rs)$,
VI)	6 "	$Y(pq)$,
VII)	the 11 functions	(pq), $Y\Delta$, Y^3, Y^2, Y, 1,

that is to say, 142 such functions. These must therefore be connected by 30 linear relations.

In the hyperelliptic case we may neglect 6 of III) which reduce to combinations of I), II) and V); and if we limit ourselves to functions of the fourth or less degree in the derivatives we may neglect IV), $Y\Delta$, Y^3. Also we may neglect VI), Y^2, Y. We thus have 120 functions of degree not greater than four in the second derivatives. They must therefore be connected by at least eight linear relations. In fact it will appear later that there are fifteen such

linearly independent relations. (These relations must of course be connected. In fact, if we use the second derivatives as coordinates in space of six dimensions, we have eight five-folds which pass through a common three-fold. This three-fold is of the eighth order.)

In the non-hyperelliptic case we may neglect IV), which may be expressed in terms of I), II), etc., by the fourth-order relation, and similarly we may neglect Y^3 and Y^2. We thus have 134 functions, and they are at most linear in Y. There must thus be 22 relations of the type $YQ_i + A_i\Delta = K_i$, where Q_i is quadratic, K_i quartic in the second derivatives and A_i is a constant. By consideration of the functions of the fifth order we can show that six of these relations must be of the form $YQ_i = C_i$, where C_i is a cubic, and the quadratic Q_i is linear in the quantities Δ_{pq}, (pq).

Again, it may easily be shown that $R_{pq}, \equiv Y_{pq} - 6(pq)Y$, is of the third order, and therefore, if the case is not hyperelliptic, R_{pq} = a linear function of Δ_{pq}, (pq).

In the hyperelliptic case it appears that this is not true; in fact R_{pq} = a linear function of Δ_{pq}, (pq), J; where J is a certain function cubic in the derivatives, and J occurs in at least one of the expressions.

We now determine the equations in detail. In the first place

$$(pqrs) - 2[(pq)(rs) + (pr)(qs) + (ps)(qr)]$$

is of the second order, and is therefore a linear function of 1, (pq), Y. We thus have fifteen equations of the type

$$(pqrs) - 2[(pq)(rs) + (pr)(qs) + (ps)(qr)] = B_{pqrs} + a_{pqrs}Y, \quad [p, q, r, s = 1, 2, 3], \tag{1}$$

where $B_{pqrs} = \Sigma\, b^{(ij)}_{pqrs}(ij) + b_{pqrs}$, the summation being taken once for each pair of values i, j, and the a's and b's are constants.

If this be differentiated with respect to the variables with suffixes t and u, we obtain

$$\begin{aligned}(pqrstu) - 2\{&(pqtu)(rs) + (pq)(rstu) + (prtu)(qs) + (pstu)(qr) \\ &+ (pr)(qstu) + (ps)(qrtu) + (pqt)(rsu) + (pqu)(rst) + (prt)(qsu) \\ &+ (pru)(qst) + (pst)(qru) + (psu)(qrt)\} = B_{pqrs,\,tu} + a_{pqrs}Y_{tu}.\end{aligned}$$

If s, t are interchanged we obtain another such equation. From these two the sixth derivative may be eliminated by subtraction. The result is

$$2\{(pqu, r)_{st} + (pru, q)_{st} + (qru, p)_{st} + (pq, ru)_{st} + (pr, qu)_{st} + (qr, pu)_{st}\}$$
$$= B_{pqrs,\,tu} - B_{pqrt,\,su} + a_{pqrs} Y_{tu} - a_{pqrt} Y_{su},$$

where

$$(pqu, r)_{st} \equiv (pqus)(rt) - (pqut)(rs),$$
$$(pq, ru)_{st} \equiv (pqs)(rut) - (pqt)(rus).$$

If we permute the suffixes $p, q, r, u,$ we obtain four such equations. They involve the three third-derivative expressions $(pq, ru)_{st}$ and certain fourth and second derivatives. By adding we eliminate the third derivatives, and then by means of (1) we may express fourth derivatives in terms of second derivatives and Y. We thus obtain the equation

$$[pqrs, ut] - [pqrt, us] + a_{pqrs} R_{ut} - a_{pqrt} R_{us} + \text{three similar expressions obtained by interchanging } u \text{ with } p \text{ and with } q \text{ and with } r = 0, \qquad (2)$$

where $[pqrs, ut] = B_{pqrs,\,ut} - 6(ut) B_{pqrs}$, in which the fourth derivatives have been replaced by their values from (1).

In addition the four equations mentioned above serve to determine the quantities $(qr, pu)_{st}$. We have in fact

$$\begin{aligned}
4(qr, pu)_{st} &+ 4\{[(uq)(sp) + (us)(pq)](rt) + [(ur)(sp) + (us)(rp)](qt) \\
&- [(uq)(tp) + (ut)(pq)](rs) - [(ur)(tp) + (ut)(rp)](qs)\} + 4Y\{(rt)a_{pqsu} \\
&- (rs)a_{pqtu} + (qt)a_{prsu} - (qs)a_{prtu} - (pt)a_{qrsu} + (ps)a_{qrtu} - (ut)a_{pqrs} \\
&+ (us)a_{pqrt}\} + 4\{(rt)B_{pqsu} - (rs)B_{pqtu} + (qt)B_{prsu} - (qs)B_{prtu} \\
&- (pt)B_{qrsu} + (ps)B_{qrtu} - (ut)B_{pqrs} + (us)B_{pqrt}\} \\
= a_{pqrs}R_{tu} &- a_{pqrt}R_{su} + a_{qrsu}R_{tp} - a_{qrtu}R_{sp} \\
&+ [pqrs, tu] - [pqrt, su] + [qrsu, tp] - [qrtu, sp]. \qquad (3)
\end{aligned}$$

It is worthy of remark that the above equations (2), (3) are of the same general form for any number of variables, for any value of p, the only difference being that if p is greater than three there is more than one function Y.

The equations (2) are linear in the quantities $R_{\alpha\beta}$, $\Delta_{\alpha\beta}$, $(\alpha\beta)$, Y. If we regard $R_{\alpha\beta}$ and Y as unknowns, there are different cases according as they can be solved for some or all of these variables or not. Suppose first that they can be solved for Y; then they give Y as a linear function of Δ_{pq}, (pq). By making

a convenient linear transformation on the fundamental variables this may be modified into one of the forms

$$Y = \Delta_{11} + \Delta_{22} + \Delta_{33}, \quad Y = \Delta_{23}, \quad Y = \Delta_{11}.$$

In this paper we neglect the second two forms, which lead to less general cases than the first, and assume that

$$Y = \Delta_{11} + \Delta_{22} + \Delta_{33}.$$

If we calculate R_{pq} directly from this value of Y, and then use (1), (3) to eliminate third and fourth derivatives, we obtain equations which show that R_{pp} can not be a linear function of the quantities Δ_{rs}, (rs), and from them we deduce without difficulty that

$$a_{pppp} = 3, \quad a_{ppqq} = 1, \quad a_{pppq} = 0, \quad a_{ppqr} = 0, \qquad (p \neq q \neq r).$$

If we multiply the 15 equations (1) by du_1^4, etc., and add, we obtain the equation

$$d^4\wp - 6(d^2\wp)^2 = B + 3C^2Y,$$

where B is a homogeneous quartic in the differentials du, with coefficients linear in the second derivatives (pq), and C is $du_1^2 + du_2^2 + du_3^2$.

When the particular values given above for the a's are substituted in the equations (2), it appears that they may be solved for the five magnitudes

$$R_{11} - R_{22}, \quad R_{11} - R_{33}, \quad R_{23}, \quad R_{31}, \quad R_{12}.$$

Accordingly each of these five must be expressible as a linear function of the quantities Δ_{pq}, (pq). We therefore assume linear functions of this type, with unknown coefficients, for these magnitudes and substitute their values and that of Y in the equations (2). These equations are now quadratics in second derivatives only, and hence they must vanish identically. Hence by equating their coefficients to zero we obtain equations, which are in fact all that exist among the undetermined constants.

Now the complete determination of the coefficients would be somewhat long if we introduced no further restriction on our choice of variables u_1, u_2, u_3. We notice, however, that Y is an algebraic invariant of the ternary forms $d^2\wp$ and C. In fact, if $d^2\wp = a_x^2 = b_x^2 = \ldots$, and $C = A_x^2 = B_x^2 = C_x^2 = \ldots$, where x is du, we have the relation $3Y(ABC)^2 = (Aab)^2$.

There is thus a certain amount of freedom at our disposal which may be used to get the equations in canonical form. We may, for example, perform any linear transformation on the variables u_1, u_2, u_3, that leaves the quadric C invariant. Also we may add constants to the second derivatives (pq), provided we subtract an appropriate linear function of the second derivatives from Y. Further, it is suggested that the equations can be modified so as to be covariants of certain ternary forms. It appears that the constants to be added to (pq) are uniquely determinate if the equations are to be covariant, and hence that the covariant canonical form is perfectly definite.

The details of the process I adopted are as follows: I first wrote out the general equations among the constants, and then proved that by a certain transformation on the u's the coefficients of type $b_{1112}^{(33)}$ and certain constants in the R's could all be made zero. When this simplification was introduced, the constants were all readily calculated in terms of six left arbitrary, and the equations were then in a canonical, though not covariant shape. It was clear, however, that by slight modification they might be made covariantive if a certain fixed quartic were associated, and the proper modification was given without much trouble by comparing terms in B that involved second derivatives with possible covariants.

The fixed forms entering into the covariant equation are the conic C and a certain quartic. The quartic is trinodal, and its nodes are at the vertices of a self-conjugate triangle of the conic. It is clear that by taking C in the form $x_1x_3 - x_2^2$, introducing a parametric pair t_1, t_2, and writing $x_1 = t_1^2$, $x_2 = t_1t_2$, $x_3 = t_2^2$, we may make use of binariants involving a single octavic, that cut out on the conic by the quartic. This leads to a set of equations equivalent to the one given by Baker in the papers already quoted, and serves to identify our equations with the hyperelliptic case, for which a binary octavic is fundamental. I prefer, however, to keep C in the form $x_1^2 + x_2^2 + x_3^2$, so as to preserve symmetry, and to work with orthogonal invariants of a quartic. The quartic is taken to be

$$6\Sigma h_1 x_2^2 x_3^2 + 12\Sigma p_1 x_1^2 x_2 x_3 \equiv \alpha_x^4 \equiv \beta_x^4 \equiv \gamma_x^4 \equiv \dots .$$

It involves only five constants, namely the ratios of the six quantities h_1, h_2, h_3, p_1, p_2, p_3. This was to be expected, since the hyperelliptic functions for $p = 3$ possess only five class-moduli. In the non-hyperelliptic case we expect six essential constants.

The values of the coefficients of B for the canonical form are the following:

$$B_{1111} = 6(h_2 + h_3)(11) + 6h_2(22) + 6h_3(33) - 12p_1(23) + b_{1111},$$
$$B_{1112} = 3p_3[(11) + (22)] - 6p_2(23) + b_{1112},$$
$$B_{1122} = (h_1 + h_2 - h_3)[(11) + (22)] - 4h_3(33) + 2p_1(23) + 2p_2(13) + b_{1122},$$
$$B_{1123} = -p_1[2(11) + 3(22) + 3(33)] - 2(h_2 + h_3 - h_1)(23) + 2p_2(12) + 2p_3(13) + b_{1123},$$

where

$$b_{1111} = 3(L + M) - 6p_1^2,$$
$$b_{1112} = 3p_3(h_2 + h_3 - h_1) - 3p_1p_2,$$
$$b_{1122} = (L - M) - (h_3^2 + 2h_1h_2 - h_1^2 - h_2^2) - 2p_3^2,$$
$$b_{1123} = -2p_1(2h_1 + h_2 + h_3) + p_2p_3,$$

and

$$4L = 2(h_2h_3 + h_3h_1 + h_1h_2) - h_1^2 - h_2^2 - h_3^2,$$
$$M = p_1^2 + p_2^2 + p_3^2,$$

and the remainder of the B's are obtained by interchanging the suffixes 1, 2, 3.

We can identify the equations with an appropriate symbolic expression by making use of the fact that orthogonal invariants consist of sums of products of symbolic expressions of the types

$$(\alpha_1^2 + \alpha_2^2 + \alpha_3^2), \quad (\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3), \quad \alpha_x, \quad (\alpha\beta\gamma),$$

and then introducing the fundamental conic to obtain the regular ternary three-rowed determinants. We thus obtain the result

$$d^4\wp - 6(d^2\wp)^2 = 6(A\alpha a)^2 a_x^2 B_x^2 + 4(AB\alpha)(ABa)\alpha_x^3 a_x - 3(ABa)^2 a_x^4 + 3YA_x^2B_x^2 + Q,$$

where

$$Q = -\tfrac{1}{8}(A\alpha\beta)^2(B\alpha\beta)^2C_x^2D_x^2 + 2(A\alpha\beta)^2(B\alpha\beta)(BC\beta)C_xD_x^2\alpha_x - \tfrac{1}{2}(AB\beta)^2(CD\beta)^2\alpha_x^4$$
$$- \tfrac{1}{4}(AB\alpha)^2(CD\beta)^2\alpha_x^2\beta_x^2 - \tfrac{5}{4}(AB\alpha)(AB\beta)(CD\alpha)(CD\beta)\alpha_x^2\beta_x^2 + 2(AB\beta)^2(CD\alpha)(CD\beta)\alpha_x^3\beta_x.$$

In the course of determining the above equations we also determine the values of R_{23}, R_{31}, R_{12}, $R_{11} - R_{22}$, $R_{11} - R_{33}$. By direct differentiation of Y we can calculate, for example, R_{11}, and thus we have expressions for the quantities $R_{pq} \equiv Y_{pq} - 6(pq)Y$ in terms of second derivatives. Now R_{11}, for example, involves the as yet undetermined third-order function J. Let

$$I = \tfrac{1}{2}(A\alpha a)^2(B\alpha b)^2, = \Sigma h_1(11)^2 + (h_1 + h_2 + h_3)\Sigma(11)(22) - 2\Sigma p_1(23)[(22) + (33)] + 4\Sigma p_1(12)(13),$$

then

$$J = 2Y[(11) + (22) + (33)] - \Delta + I,$$

and

$$\begin{aligned}R_{11} = & -4J - 4(h_1+h_2+h_3)(\Delta_{22}+\Delta_{33}) - 8h_1\Delta_{11} + 8p_2\Delta_{31} + 8p_3\Delta_{12} \\ & + [2p_1^2 + 8p_2^2 + 8p_3^2 - 14L + 3(h_1^2 + 2h_2h_3 - h_2^2 - h_3^2)](11) \\ & + [8p_1^2 + 4p_2^2 + 14p_3^2 - 8L + (h_2^2 + 2h_3h_1 - h_3^2 - h_1^2)](22) \\ & + [8p_1^2 + 14p_2^2 + 4p_3^2 - 8L + (h_3^2 + 2h_1h_2 - h_1^2 - h_2^2)](33) \\ & - [22p_2p_3 - 4p_1(4h_1 - h_2 - h_3)](23) - [14p_3p_1 + 2p_2(7h_1 - h_2 + h_3)](31) \\ & - [14p_1p_2 + 2p_3(7h_1 + h_2 - h_3)](12) - (h_2+h_3-h_1)(h_3+h_1-h_2)(h_1+h_2-h_3) \\ & + 2p_1^2(7h_1 + 2h_2 + 2h_3) + 2p_2^2(2h_1 + 7h_2 + 2h_3) + 2p_3^2(2h_1 + 2h_2 + 7h_3) \\ & - 16p_1p_2p_3.\end{aligned}$$

Also

$$\begin{aligned}R_{12} = & \, 4p_3(\Delta_{11}+\Delta_{22}) - 8p_1\Delta_{13} - 8p_2\Delta_{23} - 8h_3\Delta_{12} \\ & + [p_3(h_2 - 5h_1 - h_3) - 2p_1p_2](11) + [p_3(h_1 - 5h_2 - h_3) - 2p_1p_2](22) \\ & + [6p_3h_3 - 6p_1p_2](33) + [4(h_1 + 2h_2 + h_3)p_2 - 2p_1p_3](23) \\ & + [4(2h_1 + h_2 + h_3)p_1 - 2p_2p_3](31) \\ & + [6(L+M) - 4p_1^2 - 4p_2^2 + 4h_3(h_3 - h_1 - h_2)](12).\end{aligned}$$

These expressions must of course be coefficients of covariants. The determination of the appropriate symbolic expressions involves rather long calculation for the lower-degree terms in second derivatives. We have however the result that

$$\begin{aligned}\tfrac{1}{6}(ABC)^2[d^2Y - 6Y(d^2\varphi)] = C\{-4J + (\alpha ab)^2(\alpha AB)^2 - (\alpha AB)^2(\alpha CD)^2 Y\} \\ + 2(Aab)(Bab)(AC\alpha)(BC\alpha)\alpha_x^2 + H + KC,\end{aligned}$$

where H is a quadratic covariant linear in the quantities (pq), and K is an invariant with constant coefficients.

We next need the expressions, which we know *a priori* to exist, for products of third derivatives as cubic functions of second derivatives. It seems difficult to find these by direct integration, as may be done for $p=2$, because the complication introduced by the function Y makes the algebra heavy. In the case of $p=2$ the cubics are minors of a four-rowed determinant. The corresponding determinantal expressions in our case are not at all obvious, for instead of a four-rowed determinant we now have an array with ten rows and twenty-four columns, all of whose ten-rowed determinants must vanish. The third derivatives are proportional to first minors of these determinants, but it would seem impossible to readily factor the nine-rowed determinants, since there are many quartic relations among their elements.

We therefore use the method indicated for $p=2$. We have, as before, the equation

$$d^6\wp - 30\,d^2\wp\, d^4\wp + 60\,(d^2\wp)^3 = 12X,$$

where X is a covariant of the sixth order, with coefficients of the first degree in (pq), Y. Also

$$d^4\wp - 6\,(d^2\wp)^2 = B + 3\,YC^2.$$

Hence, by differentiation

$$d^6\wp - 12\,(d^2\wp)(d^4\wp) - 12\,(d^3\wp)^2 = d^2B - 3\,C^2 d^2 Y,$$

and therefore, by elimination of $d^6\wp$, $d^4\wp$, we have the equation

$$(d^3\wp)^2 - 4\,(d^2\wp)^3 - (B + 3\,YC^2)\,d^2\wp + \tfrac{1}{12}[d^2B - 6\,B(d^2\wp)] \\ + \tfrac{1}{4}C^2\,[d^2Y - 6\,Y(d^2\wp)] = X. \qquad (4)$$

The terms of the second degree in $d^2B - 6B(d^2\wp)$ may be readily obtained from the symbolic expression for B. They are all linear functions of the quantities Δ_{pq}, (pq), and hence are only of the third order. Also the quantities $d^2Y - 6\,Y(d^2\wp)$ are known, so that the coefficients of the left-hand side of (4) are easily obtained. We can therefore express such functions as (111), (111)(112), $2(111)(122) + 3(112)^2$, etc., by means of quantities of the third degree in second derivatives, and all the coefficients except those arising from X are known. Again, (3) gives, for example, $(111)(122) - (112)^2$ as a third-degree expression, and hence we can find (111)(122) and $(112)^2$. Similarly we can find expressions for all the remaining products of third derivatives, though the coefficients arising from X are as yet undetermined. Now $Y = \Delta_{11} + \Delta_{22} + \Delta_{33}$ and therefore

$$Y_1 = (111)[(22) + (33)] + (122)[(33) + (11)] + (133)[(11) + (22)] \\ - 2(123)(23) - 2(113)(13) - 2(112)(12).$$

We multiply this equation by any third derivative, and substitute the expressions already obtained for products of third derivatives. We thus obtain the values of such products as $(pqr)\,Y_s$ in terms of second derivatives, and in these expressions all the coefficients are known except those of the second and lower degrees and those associated with such functions as $(pq)\,Y$.

We now obtain, by differentiation and subtraction of the fundamental equations (1), 24 equations linear in second derivatives, linear in third derivatives, and possibly containing a term that is a first derivative of Y.

For example, we have

$$2(111)(12) - 2(112)(11) = 2\,(h_2 + h_3)(112) + 2\,h_2(222) + 2\,h_3(233) \\ - 4p_1(223) - p_3[(111) + (122)] + 2p_2(123) + Y_1.$$

We multiply these equations by one of the third derivatives, substitute for products of third derivatives, and obtain finally an equation which is either of the fourth or of the second degree in second derivatives. The fact that the equations of the second degree must vanish identically enables us to determine the unknown constants, and then the remaining equations give the relations of the fourth degree among the second derivatives.

The work indicated above is very long, both on account of the number of functions to be calculated, and on account of the magnitude of the final results. We content ourselves therefore, for the present, with giving the terms of highest degree for the various functions mentioned. The typical products of third derivatives are given by the equations

$$\begin{aligned}
&(111)^2 - 4(11)^3 - J - 3(11)Y = \ldots\ldots, \\
&(111)(112) - 4(11)^2(12) - (12)Y = \ldots\ldots, \\
&(111)(122) - 2(11)^2(22) - 2(11)(12)^2 + J - [(11) + 2(22)]\,Y = \ldots\ldots, \\
&(112)^2 - 4(11)(12)^2 + Y(22) - J = \ldots\ldots, \\
&(111)(123) - 2(11)(12)(13) - 2(11)^2(23) - 2Y(23) = \ldots\ldots, \\
&(112)(113) - 4(11)(12)(13) + Y(23) = \ldots\ldots, \\
&(111)(222) + 2(12)^3 - 6(11)(22)(12) + 3Y(12) = \ldots\ldots, \\
&(112)(122) - 2(11)(22)(12) - 2(12)^3 - Y(12) = \ldots\ldots, \\
&(111)(223) - 4(11)(12)(23) - 2(11)(13)(22) + 2(12)^2(13) + Y(13) = \ldots\ldots, \\
&(112)(123) - 2(11)(12)(23) - 2(12)^2(13) = \ldots\ldots, \\
&(113)(122) - 2(11)(13)(22) - 2(12)^2(13) - Y(13) = \ldots\ldots, \\
&(123)^2 - 4(23)(31)(12) + \Delta - Y[(11) + (22) + (33)] = \ldots\ldots, \\
&(112)(233) - 2(11)(23)^2 - 2(33)(12)^2 - \Delta + Y(22) = \ldots\ldots,
\end{aligned}$$

where the terms omitted are of the second or lower degree in the second derivatives.

Those for the products $(pqr)\,Y_s$ are

$$\begin{aligned}
(111)\,Y_1 &= Y[4(11)^2 + 4\Delta_{11} + 2\Delta_{22} + 2\Delta_{33}] + \ldots\ldots, \\
(222)\,Y_1 &= Y[4(22)(12) - 2\Delta_{12}] + \ldots\ldots, \\
(112)\,Y_1 &= Y[4(11)(12) + 2\Delta_{12}] + \ldots\ldots, \\
(122)\,Y_1 &= 2Y[(11)(22) + (12)^2 + \Delta_{22}] + \ldots\ldots, \\
(123)\,Y_1 &= 4Y(12)(13) + \ldots\ldots, \\
(223)\,Y_1 &= 2Y[(13)(22) + (12)(23)] + \ldots\ldots,
\end{aligned}$$

where the omitted terms are of the third or lower degree.

Finally, the quartics are

$$
\begin{aligned}
&(11)J + Y\Delta_{11} + \ldots\ldots = 0, \\
&(12)J + Y\Delta_{12} + \ldots\ldots = 0, \\
&(\Delta_{11} + \Delta_{22})^2 + [(11) + (22)]^2 Y + \ldots\ldots = 0, \\
&\Delta_{23}^2 + [(11) + (22)][(11) + (33)]Y + \ldots\ldots = 0, \\
&\Delta_{13}\Delta_{23} - (12)[(11) + (22)]Y + \ldots\ldots = 0, \\
&\Delta_{23}(\Delta_{11} + \Delta_{22}) + (23)[(11) + (22)]Y + \ldots\ldots = 0,
\end{aligned}
$$

where the terms omitted are of the third or lower degree.

These quartics may obviously be expressed as the coefficients of two covariants, one of the fourth, the other of the second order. It may be proved that of the 21 quartics indicated, 15 are linearly independent. *There are thus 15 linearly independent relations of the fourth degree among the second derivatives of the logarithm of a hyperelliptic theta function of genus three.*

Now these relations can not be functionally independent; it is in fact clear that since the second derivatives are functions of three independent variables, the quartics, regarded as five-fold spreads in space of six dimensions, must all pass through a common three-fold. It is easy to see, by consideration of the highest-degree terms, that the surface at infinity for this three-fold is given by the vanishing of all the first minors of Δ. These six quantities all vanish if any three of them are zero, and it follows that the surface at infinity and therefore the three-fold itself are of the eighth degree.

Thus the generalization of the Kummer Quartic Surface is a certain three-fold spread of the eighth degree in space of six dimensions.

§ 4.

We next consider the non-hyperelliptic case. The function Y is not now a quadratic function of the second derivatives, and thus the third-order functions Δ_{pq}, (pq), Y, 1 are linearly independent. Hence there can be no others linearly independent of these, and therefore the quantities R_{pq} must be linearly expressible in terms of them. We write

$$R_{pq} = \sum_{h,k} c_{pq,hk}\,\Delta_{hk} + \sum_{h,k} d_{pq}^{(hk)}\,(hk) + d_{pq} + e_{pq}\,Y,$$

where the summation extends once to each pair of values of h, k, and substitute in the equation (2). This equation now becomes a linear function of the fourteen third-order functions, and hence it must vanish identically.

By equating its coefficients to zero there is obtained as before a set of relations among the various constants involved. It appears that there are no limitations on the constants a_{pqrs}. We may modify our theta function as before by multiplying by an exponential factor, and also we may add any linear function of the second derivatives to Y. The only additional constants that enter into the equation may be got rid of by these modifications, and therefore the only essential constants are the a's. We find that the modifications above mentioned can be made in only one way (with a trivial exception), if the equations are to be covariantive. In this case there is one fixed form, which is the general quartic

$$F \equiv \alpha_x^4 \equiv \beta_x^4 \equiv \dots \equiv \sum_{p,q,r,s} a_{pqrs} x_p x_q x_r x_s, \qquad (p, q, r, s = 1, 2, 3).$$

The equations, in symbolic form, are given by

$$d^4\wp - 6(d^2\wp)^2 = YF + (a\alpha\beta)^2 \alpha_x^2 \beta_x^2 - \tfrac{1}{18} S, \tag{5}$$

where

$$S = (\beta\gamma\delta)(\gamma\delta\alpha)(\delta\alpha\beta)(\alpha\beta\gamma)\,\alpha_x \beta_x \gamma_x \delta_x;$$

and

$$d^2 Y - 6Yd^2\wp = -2(ab\alpha)^2 \alpha_x^2 + \tfrac{1}{6}(\alpha\beta a)^2 (\alpha\beta\gamma)^2 \gamma_x^2 \\ - \tfrac{1}{72}(\alpha\beta\gamma)^2 (\alpha\delta\varepsilon)^2 (\delta\beta\gamma)(\varepsilon\beta\gamma)\,\delta_x \varepsilon_x. \tag{6}$$

These covariant expressions are definite except in one particular. It is seen at once that Y behaves like an invariant of the third degree. Now $A \equiv \frac{1}{6}(\alpha\beta\gamma)^4$ is a similar invariant, and hence we may, if we like, take instead of Y the function $Y' = Y + \lambda A$, where λ is an absolute constant. There is then a corresponding modification to be made to the second term on the right-hand side of (6). For instance, if $Y' = Y - \frac{1}{18} A$, the identity

$$(\alpha\beta\gamma)^2 (\beta\gamma a)^2 \alpha_x^2 = 2(\alpha\beta a)(\alpha\gamma a)(\alpha\beta\gamma)^2 \beta_x \gamma_x + 2A a_x^2$$

shows that this second term becomes $\frac{1}{3}(\alpha\beta a)(\alpha\gamma a)\beta_x\gamma_x$.

These expressions were obtained by taking the quartic in the canonical form

$$F = x_1^4 + x_2^4 + x_3^4 + 6\Sigma h_1 x_2^2 x_3^2 + 12\Sigma p_1 x_1^2 x_2 x_3,$$

and calculating the constants directly from the equations among the coefficients of the equations of type (2). It then appeared that the part of B (we recall that B is used to denote that part of the right-hand side of (5) that does not involve Y) involving first derivatives was quadratic in the constants of F. The only available covariant was therefore $(\alpha\beta a)^2 \alpha_x^2 \beta_x^2$, and it was found that by adding

suitable constants to the second derivatives, and incorporating a linear function of these second derivatives into Y, this part of B could be identified with this covariant. It is interesting to note that the method followed leads to the explicit forms of the various covariants involved, and therefore is one for calculating certain covariants of the quartic. The constant terms are two such covariants for which we thus have explicit expressions. They are both given by Salmon* for the particular case in which p_1, p_2, p_3 are all zero.

We proceed to give the explicit forms of the equations:

$$\begin{aligned}
B_{1111} &= 2(h_2h_3 - p_1^2)(11) + 2h_2(22) + 2h_3(33) - 4p_1(23) + b_{1111},\\
B_{1112} &= (h_3p_3 - 2p_1p_2)(11) + p_3(22) - 2(h_2h_3 - p_1^2)(12) - 2p_2(23) + b_{1112},\\
B_{1122} &= \tfrac{1}{3}(h_2 + h_1h_3 - 4p_2^2)(11) + \tfrac{1}{3}(h_1 + h_2h_3 - 4p_1^2)(22) + \tfrac{1}{3}(1 - 3h_3^2)(33)\\
&\quad + 2h_3p_1(23) + 2h_3p_2(31) + \tfrac{2}{3}(5p_1p_2 - 4h_3p_3)(12) + b_{1122},\\
B_{1123} &= -\tfrac{2}{3}(h_1p_1 + p_2p_3)(11) - h_2p_1(22) - h_3p_1(33) + \tfrac{2}{3}(2h_2h_3 + p_1^2 - h_1)(23)\\
&\quad + \tfrac{2}{3}(h_3p_3 + p_1p_2)(31) + \tfrac{2}{3}(h_2p_2 + p_3p_1)(12) + b_{1123},
\end{aligned}$$

where the coefficients of $-S$ are given by

$$\begin{aligned}
18b_{1111} &= -24(h_2h_3 - p_1^2) - 24p_1p_2p_3 + 24h_2p_2^2 + 24h_3p_3^2,\\
18b_{1112} &= 6\{(h_3p_3 - 2p_1p_2)(p_1^2 - h_2h_3) + 2h_3h_1p_3 - h_1p_1p_2 - h_2p_3\},\\
18b_{1122} &= 4h_1h_2h_3^2 - 4h_1h_3p_1^2 - 4h_2h_3p_2^2 - 4h_3^2p_3^2 + 16h_3p_1p_2p_3 - 12p_1^2p_2^2\\
&\quad - 4h_1p_2^2 - 4h_2p_1^2 + 4h_1^2h_3 + 4h_2^2h_3 - 4h_1h_2 - 4p_3^2,\\
18b_{1123} &= -8h_1h_2h_3p_1 - 6h_2h_3p_2p_3 + 8h_1p_1^3 + 8h_2p_1p_2^2 + 8h_3p_1p_3^2 - 14p_1^2p_2p_3\\
&\quad + 4h_1p_2p_3 - 2h_1^2p_1 + 2p_1.
\end{aligned}$$

Also

$$\begin{aligned}
R_{11} &= -4\Delta_{11} - 4h_3\Delta_{22} - 4h_2\Delta_{33} - 8p_1\Delta_{23}\\
&\quad + \tfrac{1}{3}[2h_1h_2h_3 - 2p_1p_2p_3 + 4h_1p_1^2 + 2h_2p_2^2 + 2h_3p_3^2 + 3h_1^2 + h_2^2 + h_3^2 + 1](11)\\
&\quad + \tfrac{1}{3}[4h_2(h_2h_3 - p_1^2) + 2(h_1h_2 + h_3 + p_3^2)](22) + (\bigstar)(33)\\
&\quad + \tfrac{1}{3}[8p_1(h_1 - h_2h_3 + p_1^2) - 2p_2p_3](23)\\
&\quad + \tfrac{1}{3}[6p_1p_3h_3 - 2p_2(h_2h_3 + 2p_1^2 + 3h_1)](31) + (\bigstar)(12) + d_{11},\\
R_{12} &= -4p_3\Delta_{33} - 8p_2\Delta_{23} - 8p_1\Delta_{31} - 8h_3\Delta_{12}\\
&\quad - \tfrac{1}{3}[5h_1h_3p_3 + 2h_1p_1p_2 - h_2p_3](11) + (\bigstar)(22)\\
&\quad + \tfrac{1}{3}[7h_3(h_3p_3 - 2p_1p_2) + p_3](33)\\
&\quad + \tfrac{1}{3}[4p_2(3h_2h_3 + p_1^2 + h_1) - 2h_3p_1p_3](23) + (\bigstar)(31)\\
&\quad + \tfrac{1}{3}[4h_1h_2h_3 - 10p_1p_2p_3 + 8h_1p_1^2 + 8h_2p_2^2 + 12h_3p_3^2 + 4h_3^2](12) + d_{12},
\end{aligned}$$

* "Higher Plane Curves," 3rd Ed., pp. 270, 273.

where

$$
\begin{aligned}
9d_{11} = {} & 4h_1h_2^2h_3^2 + 4h_1h_2h_3p_1^2 - 8h_1p_1^4 + 8p_1^3p_2p_3 + 28h_2h_3p_1p_2p_3 - 8h_2p_1^2p_2^2 \\
& - 10h_2^2h_3p_2^2 - 8h_3p_1^2p_3^2 - 10h_2h_3^2p_3^2 - 6h_1^2h_2h_3 - 20h_1p_1p_2p_3 + 6h_1^2p_1^2 \\
& + 8h_1h_2p_2^2 + 8h_1h_3p_3^2 + 6p_2^2p_3^2 - 2h_2^2p_1^2 - 2h_3^2p_1^2 + 2h_2^3h_3 + 2h_2h_3^3 \\
& + 2h_2p_3^2 + 2h_3p_2^2 - 2h_2h_3 + 2p_1^2, \\
9d_{12} = {} & - 8h_1h_2h_3^2p_3 - 2h_1h_2h_3p_1p_2 - 6h_3p_1p_2p_3^2 + 4p_1^2p_2^2p_3 + 2h_1h_3p_1^2p_3 \\
& - 4h_1p_1^3p_2 + 2h_2h_3p_2^2p_3 - 4h_2p_1p_2^3 + 2h_3^2p_3^3 + 2h_1^2h_3p_3 + 2h_2^2h_3p_3 \\
& + 2h_1^2p_1p_2 + 2h_2^2p_1p_2 + 2h_3^3p_3 - 4h_3^2p_1p_2 - 6h_1p_2^2p_3 - 6h_2p_1^2p_3 - 2p_3^3 \\
& + 4h_1h_2p_3 - 2h_3p_3 + 2p_1p_2.
\end{aligned}
$$

The expressions omitted are obtained by appropriate interchange of the suffixes 1, 2, 3.

We shall find it convenient to use the symbolic notation for most of the remainder of our work. We use $p_x^3 \equiv q_x^3 \equiv \ldots$ for the form whose coefficients are third derivatives, and ξ_x^4 for fourth derivatives; also L_x, M_x^2, N_x^3, ... are used to denote first, second, etc., derivatives of Y.

It is easy to differentiate a symbolic expression. For example, suppose we have a covariant linear in second derivatives, say $(a\alpha\beta)^2\alpha_x^2\beta_x^2$. The first derivative of this is $(p\alpha\beta)^2p_x\alpha_x^2\beta_x^2$; the second is

$$(\xi\alpha\beta)^2\alpha_x^2\beta_x^2\xi_x^2 \equiv \xi_y^2\xi_x^2\alpha_x^2\beta_x^2.$$

We now use the polarized form of (5) and can substitute for ξ at once.

The differential coefficient of $(ab\alpha)^2\alpha_x^2$ is $2(pa\alpha)^2p_x\alpha_x^2$. Its second derivative is

$$2(pq\alpha)^2p_xq_x\alpha_x^2 + 2(\xi a\alpha)^2\xi_x^2\alpha_x^2.$$

This may be expressed in terms of second derivatives and Y only, as soon as we know the expression for $(pqu)^2p_xq_x$, that is to say, the expression for (3) in symbols.

We now proceed to get some equations that we need in symbolic form. We first take (5) in completely polarized form, $d^{(1)}d^{(2)}d^{(3)}d^{(4)}\varphi = \ldots$, and perform on it the operation $d^{(5)}$. We then interchange $d^{(4)}$ and $d^{(5)}$ and subtract. We thus obtain an equation which is linear in second derivatives, linear in third derivatives, and linear in first derivatives of Y. This is the polarized form of

$$-6(pau)p_x^2a_x = (p\alpha\beta)^2(p\beta u)\alpha_x^2\beta_x + (Lau)\alpha_x^3, \tag{7}$$

where u has been written for $(x^{(4)}x^{(5)})$. In exactly the same way we derive from (7) the equation

$$4(pqu)^2 p_x q_x = 2(\xi a u)^2 \xi_x^2 + \tfrac{1}{3}(\xi\alpha\beta)^2(\xi\beta u)^2\alpha_x^2 + \tfrac{2}{3}(\xi\alpha\beta)^2(\xi\alpha u)(\xi\beta u)\alpha_x\beta_x + (M\alpha u)^2\alpha_x^2. \tag{8}$$

If in this we replace fourth derivatives by their values in terms of second derivatives, and substitute from (6) for second derivatives of Y, we have equation (3) given in symbolic form.

We shall need in later work the value of the second derivative of $(abu)(abv)$. This may be obtained by polarizing from $(abu)^2$, and so we give $d^2(abu)^2$.

Now

$$d^2(abu)^2 = 2(pqu)^2 p_x q_x + 2(\xi a u)^2 \xi_x^2,$$

and therefore from (8)

$$2d^2(abu)^2 = 6(\xi a u)^2 \xi_x^2 + \tfrac{1}{3}(\xi\alpha\beta)^2(\xi\beta u)^2\alpha_x^2 + \tfrac{2}{3}(\xi\alpha\beta)^2(\xi\alpha u)(\xi\beta u)\alpha_x\beta_x + (M\alpha u)^2\alpha_x^2.$$

§ 5.

Exactly as in previous cases we have an equation

$$d^6\wp - 30\,d^2\wp\, d^4\wp + 60(d^2\wp)^3 = X + hY,$$

where X consists of two parts, one linear in second derivatives, and the other constant, whilst h is constant, both X and h being sextic covariants. As before, we eliminate fourth and sixth derivatives by means of (5) and thus obtain an equation

$$12\,[(d^3\wp)^2 - 4(d^2\wp)^3 - (d^2\wp)(B + FY)] + K = X + hY, \tag{9}$$

where

$$K = d^2B - 6Bd^2\wp + F(d^2Y - 6Yd^2\wp).$$

Now

$$B = (a\alpha\beta)^2\alpha_x^2\beta_x^2 + \text{const.},$$

therefore

$$\begin{aligned} d^2B &= (\xi\alpha\beta)^2\alpha_x^2\beta_x^2\xi_x^2 \\ &= 2(a\alpha\beta)^2\alpha_x^2\beta_x^2 b_x^2 + 4(a\alpha\beta)(b\alpha\beta)a_x b_x \alpha_x^2\beta_x^2 + \tfrac{1}{3}(a\gamma\delta)^2(\alpha\beta\gamma)^2\alpha_x^2\beta_x^2\delta_x^2 \\ &\quad + \tfrac{2}{3}(a\gamma\delta)^2(\alpha\beta\gamma)(\alpha\beta\delta)\alpha_x^2\beta_x^2\gamma_x\delta_x + (\sigma\alpha\beta)^2\sigma_x^2\alpha_x^2\beta_x^2 + (\alpha\beta\gamma)^2\alpha_x^2\beta_x^2\gamma_x^2\,Y; \end{aligned}$$

and

$$6\,Bd^2\wp = 6(a\alpha\beta)^2\alpha_x^2\beta_x^2 b_x^2 + 6\,\sigma_x^4\alpha_x^2;$$

also

$$d^2Y - 6\,Yd^2\wp = -2(ab\alpha)^2\alpha_x^2 + \tfrac{1}{6}(a\alpha\beta)^2(\alpha\beta\gamma)^2\gamma_x^2 + \tau_x^2,$$

where

$$\sigma_x^4 = -\tfrac{1}{18} S,\ \tau_x^2 = \text{the constant term in (6)}.$$

Hence

$$\begin{aligned} K = &-6(ab\alpha)^2\alpha_x^2 F + 4(ab\alpha)(ab\beta)\alpha_x^3\beta_x^3 + [\tfrac{1}{3}(a\gamma\delta)^2(\alpha\beta\gamma)^2\alpha_x^2\beta_x^2\delta_x^2 \\ &+ \tfrac{2}{3}(a\gamma\delta)^2(\alpha\beta\gamma)(\alpha\beta\delta)\alpha_x^2\beta_x^2\gamma_x\delta_x + \tfrac{1}{3}Sa_x^2 + \tfrac{1}{6}(a\alpha\beta)^2(\alpha\beta\gamma)^2\gamma_x^2 F] \\ &+ [\sigma\alpha\beta)^2\sigma_x^2\alpha_x^2\beta_x^2 + \tau_x^2 F] + Y(\alpha\beta\gamma)^2\alpha_x^2\beta_x^2\gamma_x^2 \\ = &\ P + Q, \text{ say,} \end{aligned}$$

where

$$P = -6(ab\alpha)^2\alpha_x^2 F + 4(ab\alpha)(ab\beta)\alpha_x^3\beta_x^3 + HY,$$

and

$$H = (\alpha\beta\gamma)^2\alpha_x^2\beta_x^2\gamma_x^2.$$

Thus when we know the quantities X and h, the equations (8), (9) serve to express products of third derivatives as cubic functions of second derivatives, and involving Y linearly. These equations, however, enable us to get the expression for Y in terms of second derivatives. To obtain this relation we differentiate (9) twice, and substitute from (8) for the third derivatives that occur. We thus obtain a relation of the form

$$(Y^2 + 2\Delta)F^2 = \Sigma\alpha_{pq,\,rs}(pq)(rs) + \Sigma\beta_{pq}(pq) + \gamma + \delta Y,$$

where the coefficients on the right side are constants. It is clear that F^2 must divide through the equation, and thus we have restrictions on X and h, which enable us to determine these quantities, with the exception of the constant term in X.

When we differentiate (9) twice and substitute for fourth derivatives, we have the equation

$$12(B + YF)^2 + d^2K - 12Kd^2\wp = d^2X + hd^2Y.$$

We can calculate d^2K in symbols. Also, X being a covariant at most linear in the coefficients of a_x^2, we see that $d^2X - 6Xd^2\wp$ is an expression linear in the quantities Δ_{pq}, (pq). When the above expression is written in expanded form, it appears that F^2 is a factor of all terms except those of the types $(pq)(rs)$, $(pq)Y$, (pq), 1. It must therefore also divide the terms of the types given. It follows at once that there are no terms of the type $(pq)Y$, and hence h is determined. A consideration of terms of the type $(pq)(rs)$ determines the part of X involving the coefficients of a_x^2. We obtain finally the result

$$\begin{aligned} 12[(d^3\wp)^2 - 4(d^2\wp)^3 - (d^2\wp)(B + FY)] + K = &-\tfrac{5}{3}HY + Z \\ &- 5(a\gamma\delta)^2(\alpha\beta\gamma)^2\alpha_x^2\beta_x^2\delta_x^2 + 6(a\gamma\delta)^2(\alpha\beta\gamma)(\alpha\beta\delta)\alpha_x^2\beta_x^2\gamma_x\delta_x - \tfrac{1}{3}Sa_x^2 \\ &+ \tfrac{10}{3}AFa_x^2 + \tfrac{5}{2}F(a\alpha\beta)^2(\alpha\beta\gamma)^2\gamma_x^2 - 4(a\alpha\beta)(\delta\alpha\beta)(\alpha\beta\gamma)^2 a_x\gamma_x^2\delta_x^3 \ \ldots\ldots \quad (10) \end{aligned}$$

where

$$H = (\alpha\beta\gamma)^2 \alpha_x^2 \beta_x^2 \gamma_x^2,$$

and Z is a sextic covariant of the quartic alone, of the sixth order in the coefficients.

This equation (10), in conjunction with (8), gives expressions for all squares and products of third derivatives as cubic functions of the second derivatives, and involving Y linearly. The constant terms, the coefficients of Z, are however as yet undetermined.

The resulting equation in the above work, when F is divided out, is an expression for Y as a function of the second derivatives. This equation is

$$Y^2 - \tfrac{1}{18} A Y = -2\Delta + \tfrac{1}{6}(a\alpha\beta)^2(b\alpha\beta)^2 + \tfrac{1}{72}(a\alpha\beta)^2(\alpha\gamma\delta)^2(\beta\gamma\delta)^2 - \tfrac{5}{7} D, \quad (11)$$

where D is the six-rowed determinant, the invariant of the quartic of the sixth degree in the coefficients, given by Salmon, "Higher Plane Curves," 3rd Ed., p. 265, and called by him B.

This equation was in fact calculated as follows: All the terms except those linear in second derivatives and the constant were determined in the manner indicated above. The remaining terms, being invariant, were necessarily of the form

$$n_1(a\alpha\beta)^2(\alpha\gamma\delta)^2(\beta\gamma\delta)^2 + n_2 A^2 + n_3 D,$$

where n_1, n_2, n_3 were merely numerical constants. n_2 was then shown to be zero by means of the particular quartic $x_1^4 + x_2^4 + x_3^4$, and the remaining two constants were found by calculating out one of the equations for Salmon's form of the quartic, $\Sigma x^4 + 6\,\Sigma h_1 x_2^2 x_3^2$.

There yet remain to be determined the relations connecting second derivatives only. Now there are 70 functions of the fifth order involving second derivatives and Y. One relation is given by (11), and since there are only 63 linearly independent fifth-order functions, it is clear that there must be six other such relations.

We proceed to obtain these relations, to show, in fact, that the coefficients of squares and products of the u's in $Y(abu)^2$ may be linearly expressed by means of the other fifth-order functions.

Equation (6) in polarized form is

$$M_x M_y - 6\,Y a_x a_y = -2(ab\alpha)^2 \alpha_x \alpha_y + \tfrac{1}{6}(a\alpha\beta)^2(\alpha\beta\gamma)^2 \gamma_x \gamma_y + \tau_x \tau_y.$$

We differentiate this with respect to z, then interchange y and z in the equation obtained and subtract; we thus get the equation

$$-6(Lau)a_x = -4(pa\alpha)^2(pau)a_x + \tfrac{1}{6}(p\alpha\beta)^2(\alpha\beta\gamma)^2(p\gamma u)\gamma_x \cdots, \qquad (12)$$

where u is written for (yz).

We next differentiate (12) with respect to y, interchange x and y, subtract, and then write $(xy) = u$. We thus have the equation

$$6(Mau)^2 = 4(\xi a\alpha)^2(\xi\alpha u)^2 + 4(pq\alpha)^2(pu\alpha)(qu\alpha) - \tfrac{1}{6}(\xi\alpha\beta)^2(\alpha\beta\gamma)^2(u\xi\gamma)^2.$$

In this equation we substitute for $(Mau)^2$ from (6), and for $(pq\alpha)^2(pu\alpha)(qu\alpha)$ from (8), after writing in (8), α for u, and $(u\alpha)$ for x. We thus have the relation

$$\begin{aligned}36Y(abu)^2 - 12(ab\alpha)^2(cu\alpha)^2 + (a\alpha\beta)^2(\alpha\beta\gamma)^2(bu\gamma)^2 + 6(\tau au)^2 \\ = 6(a\xi\alpha)^2(u\xi\alpha)^2 + (\xi\alpha\beta)^2(\xi\beta\gamma)^2(a\gamma u)^2 - \tfrac{1}{2}(\xi\alpha\beta)^2(\alpha\beta\gamma)^2(\xi\gamma u)^2 \\ + (M\alpha\beta)^2(\alpha\beta u)^2.\end{aligned}$$

This relation, after substitution for ξ and M, becomes

$$\begin{aligned}&36Y[(abu)^2 - \tfrac{1}{3}(a\alpha\beta)^2(\alpha\beta u)^2 - \tfrac{1}{72}(\alpha\beta\gamma)^2(\alpha\beta\delta)^2(\gamma\delta u)^2] \\ &= 36(ab\alpha)^2(c\alpha u)^2 + [12(a\alpha\beta)^2(b\beta\gamma)^2(\gamma au)^2 - 6(\alpha\beta\gamma)^2(b\alpha\beta)^2(a\gamma u)^2 \\ &\quad - 3(ab\gamma)^2(\alpha\beta\gamma)^2(\alpha\beta u)^2 + 2A(abu)^2] + [-6(a\tau u)^2 + 6(a\sigma\alpha)^2(u\sigma\alpha)^2 \\ &\quad + (a\delta\varepsilon)^2(\delta\alpha\beta)^2(\varepsilon\beta\gamma)^2(a\gamma u)^2 - \tfrac{1}{2}(a\delta\varepsilon)^2(\delta\alpha\beta)^2(\alpha\beta\gamma)^2(\varepsilon\gamma u)^2 \\ &\quad + \tfrac{1}{3}A(a\alpha\beta)^2(\alpha\beta u)^2 - \tfrac{1}{3}(a\delta\varepsilon)^2(\alpha\delta\varepsilon)^2(\alpha\beta\gamma)^2(\beta\gamma u)^2] \\ &\quad + [(\sigma\alpha\beta)^2(\sigma\beta\gamma)^2(a\gamma u)^2 - \tfrac{1}{2}(\sigma\alpha\beta)^2(\alpha\beta\gamma)^2(\sigma\gamma u)^2 + (\tau\alpha\beta)^2(\alpha\beta u)^2] \ldots. (13)\end{aligned}$$

Again, since Y is of the second order, we may write it $\dfrac{\Phi}{\theta^2}$, where Φ is an integral function, and $\wp$ is $-\log\theta$. Hence

$$dY = -\frac{2\Phi\theta'}{\theta^3} + \frac{\Phi'}{\theta^2}, \quad d^2Y - 6Y(d^2\wp) = -4\frac{\Phi'\theta' - \Phi\theta''}{\theta^3} + \ldots.$$

Also

$$dYd^3\wp - 4Y(d^2\wp)^2 = -2\frac{\Phi'\theta' - \Phi\theta''}{\theta^3}\frac{\theta'^2}{\theta^2} + \ldots.$$

and therefore

$$dYd^3\wp - 4Y(d^2\wp)^2 - (d^2Y - 6Yd^2\wp),$$

though apparently of the sixth, is really of the fourth order. Hence

$$dYd^3\wp - 4Y(d^2\wp)^2 + (ab\alpha)^2\alpha_x^2(d^2\wp)$$

is of the fourth order. The expression of this in terms of the fourth-order functions may now be computed, and we have the result

$$
\begin{aligned}
dYd^3\wp - 4Y(d^2\wp)^2 + (ab\alpha)^2\alpha_x^2(d^2\wp) = \tfrac{1}{3}(a\alpha\beta)^2(b\beta\gamma)^2 \\
+ \tfrac{2}{9}(a\alpha\beta)^2(b\alpha\beta)(\alpha\beta\gamma)b_x\gamma_x^3 - \tfrac{4}{3}\Delta F - \tfrac{1}{18}(a\alpha\beta)^2(b\alpha\beta)^2F \\
+ \tfrac{2}{3}Y(a\alpha\beta)^2\alpha_x^2\beta_x^2 + \text{an expression of the third order.} \qquad (14)
\end{aligned}
$$

Also, if in (7) we replace u by q, and multiply by q_x^2, we have

$$(p\alpha\beta)^2(p\beta q)\alpha_x^2\beta_x p_x^2 + (L\alpha p)\alpha_x^3 p_x^2 = 0,$$

or

$$(pq\beta)^2(p\alpha\beta)\alpha_x^3\beta_x q_x + (Lp\alpha)p_x^2\alpha_x^3 = 0.$$

From this last we readily deduce

$$(Lpu)p_x^2 = (pq\alpha)^2(p\alpha u)q_x\alpha_x. \qquad (15)$$

By means of (8), the right-hand side of (15) may be expressed in terms of functions of the second order, and then (14) and (15) enable us to express any proJuct of a first derivative of Y and a third derivative of $\wp$ as a cubic function of the functions of the second order.

We now obtain two equations from (8) in exactly the same way that (7) and (8) were determined from (5). These two equations are

$$6(\xi qu)^3\xi_x = (\eta\alpha\beta)^2(\eta\beta u)^2(\eta\alpha u)\alpha_x + (N\alpha u)^3\alpha_x, \qquad (16)$$

$$-6(\xi\xi'u)^4 = (\theta\alpha\beta)^2(\theta\alpha u)^2(\theta\beta u)^2 + (P\alpha u)^4, \qquad (17)$$

where

$$\eta_x^5 = d^5\wp, \quad \theta_x^6 = d^6\wp, \quad P_x^4 = d^4Y.$$

When (17) is expanded it becomes

$$
\begin{aligned}
(abu)^2(cdu)^2 + \tfrac{3}{2}Y(a\alpha u)^2(b\alpha u)^2 + \tfrac{3}{2}(c\alpha\beta)^2(a\alpha u)^2(b\beta u)^2 + \tfrac{5}{36}\Delta(\alpha\beta u)^4 \\
- \tfrac{1}{3}(abu)^2(c\alpha\beta)^2(\alpha\beta u)^2 - \tfrac{2}{3}(ab\alpha)^2(\alpha\beta u)^2(c\beta u)^2 = \ldots . \qquad (18)
\end{aligned}
$$

where the terms on the right are at most quadratic in the second order functions.

Now we showed originally that there were 22 relations among the functions of the sixth order that were rational functions of the second derivatives and at most linear in Y. The equation (18) gives 15 of these relations, by equating to zero the various coefficients of powers and products of the u's. There are six others given similarly by equation (13). Also if in (13) we replace u by c, we have a relation independent of those already enumerated, which expresses $Y\Delta$ in terms of second derivatives. We have thus altogether 22. It is almost demonstrable that there are no others, and in fact that all other relations among second and third derivatives, involving Y and its first derivatives, may be derived

by algebraic processes from those given. In particular, there are relations among the second derivatives only. There are apparently none of order so low as 4. They may be obtained by eliminating Y from the 22 relations mentioned. For example, we obtain 15 quintic relations among second derivatives only, by elimination of Y from the equations derived from (13). Similarly we may obtain from (18) and (13) together 201 sextics, though these are not necessarily linearly independent. We notice, however, that the highest-degree terms of all these relations are homogeneous quadratics in the quantities Δ_{pq}, with coefficients quadratic functions of second derivatives. It follows that if we take second derivatives as coordinates in space of six dimensions, all the five-folds mentioned pass doubly through the surface of the eighth order at infinity given by the vanishing of all the first minors of Δ. Now we know *a priori* that these relations must have a common three-fold, and it is clear therefore that this three-fold must be either of the eighth or of the sixteenth order; it seems highly probable that it is of the sixteenth order, and the space at infinity is a trope. This three-fold is the generalization of the Kummer Quartic, which arises when $p = 2$, or of the non-singular cubic curve, for $p = 1$. We propose to consider it more in detail later.

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